A Simple Model for Pricing Securities with Equity, Interest-Rate, and Default Risk[1]

Sanjiv R. Das
Santa Clara University
Santa Clara, CA 95053

Rangarajan K. Sundaram
New York University
New York, NY 10012

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[1]The authors' respective e-mail addresses are srdas@scu.edu, and rsundara@stern.nyu.edu. We thank Santhosh Bandreddi for able research assistance, insights and comments. We are very grateful to Suresh Sundaresan, Mehrdad Noorani, an anonymous referee, and editor David Hsieh for their comments. We would also like to thank Peter Carr, Jan Ericsson, Kian Esteghamat, Rong Fan, Gary Geng, Robert Jarrow, Maxime Popineau, Taras Smetaniouk, and Stuart Turnbull; and seminar particpants at the following institutions and conferences: York University, New York University, University of Oklahoma, Stanford University, Santa Clara University, Universita La Sapienza, CreditMetrics, MKMV Corp., the 2003 European Finance Association Meetings in Glasgow, Morgan Stanley, Archeus Capital Management, Lehman Brothers, University of Massachusetts, Amherst, Derivatives Securities Conference, Washington 2005. A special thanks to CreditMetrics for the use of data.

Abstract

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We develop a model for pricing securities whose value may depend simultaneously on equity, interest-rate, and default risks. The framework may also be used to extract probabilities of default (PD) functions from market data. Our approach is based entirely on observables such as equity prices and interest rates, rather than on unobservable processes such as firm value. The model stitches together in an arbitrage-free setting a CEV equity model (to represent the behavior of equity prices prior to default), a default intensity process, and a Heath-Jarrow-Morton model for the evolution of riskless interest rates. The model captures several stylized features such as a negative relation between equity prices and equity volatility, a negative relation between default intensity and equity prices, and a positive relationship between default intensity and equity volatility. We embed the model on a discrete-time, recombining lattice, making implementation feasible with polynomial complexity. We demonstrate the simplicity of calibrating the model to market data, and of using it to extract default information. The framework is extensible to handling correlated default risk and may be used to value distressed convertible bonds, debt-equity swaps, and credit portfolio products such as CDOs. Applied to the CDX INDU Index, we find the S&P500 index explains credit premia.

1 Introduction

Several financial securities depend on more than just one category of risk. Prominent among these are corporate bonds (which depend on interest rate risk and on credit risk of the issuing firm) and convertible bonds (which depend, in addition, on equity risk). In this paper, we develop and implement a simple model for the pricing of securities whose values may depend on one or more of three sources of risk: equity risk, credit risk, and interest-rate risk.

Our framework is based on generalizing the reduced-form approach to credit-risk (Duffie and Singleton (1999), Madan and Unal (2000)) to include a process for equity. The typical reduced-form model involves two components, one describing the evolution of (riskless) interest rates, and the other, an intensity process that captures the likelihood of firm default; equity is not modeled explicitly. But any default process for a company's debt must obviously also apply to that company's equity. That is, when debt is in default, equity must also go into some post-default value. Motivated by this, we knit an equity process into a reduced-form model in an arbitrage-free manner; equity in the integrated model now follows a "jump-to-default" process, i.e., it gets absorbed at zero when a default happens.[1] The resulting framework captures simultaneously the three sources of risk mentioned above, and can be calibrated to market data to extract default probabilities or price hybrid securities.

While our model is anchored in the reduced-form approach, the specifics draw upon insights gained from the structural approach to credit risk (cf. Merton (1974), Black and Cox (1976), and others). Our starting point, the idea that default is associated with an absorbing value for equity, is itself borrowed from structural models. The process we posit for the evolution of equity prices prior to default—a constant elasticity of variance (CEV) process—is also motivated by structural models. An important characteristic of the Merton (1974) model is its generation of the so-called "leverage effect," a negative relationship between equity prices and equity volatility. The leverage effect has also been documented empirically (e.g., Christie (1982)). The CEV specification for equity prices generates a leverage effect in our reduced-form setting. Finally, we take the default intensity in our model to vary inversely with equity prices (and, therefore, directly with equity volatility). This specification too is motivated by the existence of a similar relation in the Merton (1974) model between default likelihood, equity prices, and equity volatility.

Our final framework, then, involves the following components. We have a CEV model describing the evolution of equity prices prior to default, an intensity process for default, and a riskless interest rate model (for which purpose we use the Heath-Jarrow-Morton (HJM) model, though any other interest rate model could also be used). The result is a single parsimonious model accounting for correlations that combines the three major sources of risk.

We implement the model in a discrete-time setting, using the Nelson and Ramaswamy (1990) approach to discretizing the CEV model. Rather than specify an exogenous process for the default

[1] As the junior-most claim of the firm, it is natural to set the value of equity in default to zero; this is also consistent with the assumption that absolute priority holds in bankruptcy. However, our model is easily modified to allow a non-zero value for equity in the event of default.

probability, we make it a dynamic function of both equity and interest rate information. This enables us to derive default probabilities as *endogenous* functions of the information on the lattice, jointly calibrated to equity prices and default spreads. As a consequence, default information in the model is extracted from *both* equity- and debt-market information rather than from just debt-market information (as in reduced-form credit-risk models) or from just equity-market information (as in structural credit-risk models). This allows valuation, in a single consistent framework, of hybrid debt-equity securities such as convertible bonds that are vulnerable to default, as well as of derivatives on interest rates, equity and credit. Our model can also serve as a basis for valuing credit portfolios where correlated default is an important source of risk. Finally, the model enables the extraction of credit risk premia.

Our framework has several antecedents and points of reference in the literature. We have already mentioned the connection to both reduced-form and structural models. "Jump-to-default" equity models, in which equity gets absorbed at zero following a default, have also been examined in Davis and Lischka (1999), Carayannopoulos and Kalimipalli (2003), and Carr and Linetsky (2006).[2] The first two papers use the Black-Scholes model for the equity price process prior to default which is a special case of the CEV model, and which does not admit the leverage effect.[3] The specification of the default intensity process in Davis-Lischka is also somewhat more restrictive than ours; their default intensity is perfectly correlated with the equity process, while we allow it to depend on both equity returns and interest-rates as also other information. Carayannopoulos and Kalimipalli (2003) use a default intensity specification similar to ours but their model does not allow for stochastic interest rates.

The Carr-Linetsky model uses a CEV specification for the equity price process and an endogenous specification of the intensity process. Their objectives differ from ours. We look to build a discrete-time "defaultable" tree with stochastic interest rates and equity prices which can be calibrated to market data and correlations, and which can be used to price hybrids such as convertibles that depend on equity risk, interest-rate risk, and default risk. Working in a continuous-time setting, but without interest-rate risk, Carr and Linetsky are able to provide explicit closed-form solutions for survival probabilities, CDS spreads, and European option prices.

Also related to our paper are the reduced-form models in Schönbucher (1998) and Das and Sundaram (2000) which study "defaultable HJM" models. These are HJM models with a default process tacked on. Our model generalizes these to also including equity processes. In particular, the Das and Sundaram (2000) model results as a special case of our framework if the equity process is switched off. Our framework may also be viewed as a generalization of Amin and Bodurtha (1995) (see also Brenner, Courtadon and Subrahamnyam (1987)). The Amin-Bodurtha model combines

[2]See also Linetsky (2004) for solutions in continuous time. Carr and Wu (2005) show how a similar model may be calibrated with options data. Other related papers in the literature include Jarrow (2001) and Takahashi, Kobayashi and Nakagawa (2001). Incorporation of equity risk into reduced form models has also been examined in Jarrow (2001) and Mamaysky (2002), but using a different approach: equity values are dervied through a posited dividend process.

[3]An earlier version of our paper too used the Black-Scholes model. Our investigation of a more general framework was motivated by comments from the referee and editor concerning the shortcomings of the Black-Scholes framework, in particular the absence of a leverage effect.

interest rate risk and equity risk (in the form of a Black-Scholes model) but does not incorporate credit risk. Since there is no default, equity in their model is necessarily infinitely-lived and never gets "absorbed" in a post-default value. Other frameworks too are nested within our model. For example, if the equity and hazard-rate processes are switched off, we obtain the HJM model, while if the interest-rate and hazard-rate processes are switched off, we obtain a discrete-time CEV tree, as described by Nelson and Ramaswamy (1990).

Our lattice design allows recombination, making the implementation of the model simple and efficient; indeed, the model is fully implementable on a spreadsheet. Unlike many earlier models, we are able to (a) price derivatives on equity and interest rates with default risk; (b) extract probabilities of default endogenously in the model; (c) provide for the risk-neutral simulation of correlated default risk in a manner consistent with no arbitrage and consistent with equity correlations (which we believe, has not been undertaken in any model so far); and (d) extract credit risk premia.

The rest of the paper proceeds as follows. In Section 2 we develop the pricing lattice in the state variables of the model in a manner that allows for additional structure to accommodate default risk. Section 3 deals with implementation issues, including a discussion of how default swaps may be used to calibrate the model for subsequent use. We show that the model may generate a wide range of spread curve shapes. Empirical calibration to markets is undertaken to evidence the ease of implementation. This section also explores the impact of default risk on embedded options within classic bond structures. Section 4 applies the model to the extraction of credit risk premiums and uses data from the Dow Jones CDX index to examine the principal components of these premia. Finally, an analysis of the model application to correlated default products is provided. Section 5 concludes by summarizing the economic and technical benefits of the model.

2 The Model

As we have noted already, the motivation for our model is simple. If the default process for a company's debt is described by a hazard rate λ (as in the standard reduced-form model approach), then λ must also apply to that company's equity. That is, when debt is in default, equity must also go into some default value. As the junior-most claim of the firm, it is natural to set the value of equity in default to zero, but our model is easily modified to allow a non-zero value for equity in the event of default.

An early model of "defaultable equity" was presented in Samuelson (1972) and is discussed in Merton (1976). We begin with a brief description of Samuelson's result, then discuss the directions in which we generalize it.

2.1 The Samuelson (1972) Model

Consider a continuous-time setting in which equity prices evolve according to a geometric Brownian motion

$$dS(t) \;=\; \mu S(t)\, dt + \sigma S(t)\, dW(t),$$

but with the added twist that equity prices could suddenly jump to zero and get absorbed there. Suppose that the jump-to-default is governed by a constant intensity Poisson process with hazard rate $\xi > 0$. This is a simple example of a "jump-to-default" equity process. Samuelson (1972) shows that the price of a call option on such equity is given by

$$C \;=\; \exp(-\xi T)\, C^{BS}[Se^{\xi T}, K, T, \sigma, r] \;=\; C^{BS}[S, K, T, \sigma, r + \xi].$$

Here, $C^{BS}(S, K, T, \sigma, r)$ is the standard Black-Scholes call option pricing function with current stock price S, option strike K, option maturity T, interest rate r, and stock volatility σ; and ξ is, of course, the default intensity. Notice that the call is priced by the Black-Scholes model with an adjusted risk-neutral interest rate $(r + \xi)$.[4]

From the perspective of a credit risk model, there are three weaknesses to this setting. One is the assumption of a constant equity volatility σ on the non-default segment. The so-called "leverage effect" suggests that equity volatility should increase as equity prices fall. Empirical support for the leverage effect is provided in several papers, such as Christie (1982). Theoretical support comes from structural models such as Merton (1974) in which equity prices and equity volatility are inversely related. Ideally, we would like the posited equity price process to incorporate this feature. A second weakness is the assumption of constant interest rates which makes the model inappropriate for studying hybrids such as convertible bonds which depend on both equity risk and interest-rate risk. The third is the assumption of a constant hazard rate. In general, one would expect the likelihood of a jump-to-default to be inversely related to firm value, increasing as firm value decreases. Equally, since equity is a strictly monotone function of firm value, we would expect hazard rates to move inversely to equity prices. Structural models exhibit the analogy of such a property with the likelihood of default and equity prices moving in opposite directions.

In the following sections, we describe a model that has these properties, as well as an interest-rate. We proceed in several steps, describing first the equity process we shall employ to capture the leverage effect, then the interest-rate model, and finally, the specification of the hazard rate function.

[4]Note too that the option price is not just the price of a non-defaultable call option, i.e. $BMS[S_0, K, T, \sigma, r]$, multiplied by the risk-neutral probability of survival $\exp(-\xi T)$. The drift of the risk-neutral equity process is also affected by the jump to default compensator (ξ). For an excellent exposition of default jump compensators, see Giesecke (2001).

2.2 The Equity Model

The first step in our model is to identify a process for decribing the movement of equity prices prior to default, in which equity prices and volatility move in opposite directions. A simple generalization of the Black-Scholes model which possesses the desired property is the constant-elasticity-of-variance or CEV model. In continuous time, the risk-neutral CEV equity process is described by

$$dS(t) \;=\; r(t)\, S(t)\, dt + \sigma S(t)^{\gamma}\, dZ(t), \tag{1}$$

where $\gamma \in (0, 1]$ is the CEV coefficient. It is easily checked that for $\gamma < 1$, this specification exhibits the leverage effect, while for $\gamma = 1$, it is just Black-Scholes geometric Brownian motion. We want to modify this process in two directions. One is to allow for default using an intensity process $\xi(t)$. The second is to discretize the resulting jump-to-default model.

Nelson and Ramaswamy (1990) show how to discretize the CEV process (1) in a recombining binomial tree. We generalize their construction so as to allow for a third branch from each node of the tree representing a jump-to-default. We describe here the branching process at a generic node t. Let the stock price at t be denoted $S(t)$ and the length of one period on the tree be h years. Let $\lambda(t)$ denote the (risk-neutral) likelihood of a jump-to-default at node t; this probability may depend on the information at node t, but to keep notation simple, we suppress these arguments. Finally, let $R(t)$ denote the gross (i.e., 1+ net, continuously compounded) one-period interest rate at node t. Then, the evolution of stock prices on the tree is given by the following:

$$S[Y_s(t)] \;=\; \begin{cases} [\sigma_s(1-\gamma)Y_s(t)]^{\frac{1}{1-\gamma}} & \text{if } Y_s(t) > 0 \\ 0 & \text{if } Y_s(t) \leq 0 \end{cases} \tag{2}$$

$$Y_s(t+h) \;=\; \begin{cases} Y_s(t) + \sqrt{h} & \text{with probability } q(t)[1-\lambda(t)] \\ Y_s(t) - \sqrt{h} & \text{with probability } [1-q(t)][1-\lambda(t)] \\ 0 & \text{with probability } \lambda(t) \end{cases} \tag{3}$$

$$q(t) \;=\; \begin{cases} \dfrac{[R(t)/(1-\lambda(t))] - b(t)}{a(t) - b(t)} & \text{if } Y_s(t) > 0 \\ 0 & \text{if } Y_s(t) \leq 0 \end{cases} \tag{4}$$

$$a(t) \;=\; \frac{S[Y_s(t) + \sqrt{h}]}{S[Y_s(t)]} \tag{5}$$

$$b(t) \;=\; \frac{S[Y_s(t) - \sqrt{h}]}{S[Y_s(t)]} \tag{6}$$

The tree described here is the risk-neutral tree, i.e., all the probabilities are risk-neutral probabilities. In the language of the usual binomial tree, $a(t)$ is the size of the "up move" in the binomial tree, and $b(t)$ the size of the "down move." On the non-default part of the tree, we may write

$$Y_s(t+h) \;=\; Y_s(t) + X_s(t)\sqrt{h}, \qquad X_s(t) \in \{+1, -1\}, \tag{7}$$

where X_s is a binomial random variable driving the equity process in the model prior to default. The representation (7) makes it easier to show how the desired correlation between the equity process in the model and the term structure of interest rates may be injected. Note that in the CEV model, even without jumps, the process for the stock price may be absorbed at zero; thus, our model has both "drift to default" and "jumps to default," i.e., default can occur from both the diffusion and jump processes.

The next segment describes the interest-rate model. Following that, we stitch together the equity and interest rate processes, and then, finally, take up the specification of the default probability $\lambda(t)$.

2.3 Term-Structure Model

We adopt the discrete-time, recombining form of the Heath, Jarrow and Morton (1990) model. We quickly review this here, before moving on to the description of the joint lattice, and readers may examine the original HJM paper for comprehensive details. Initially, we prepare the univariate HJM lattice for the evolution of the term structure, and subsequently stitch on the equity process defined above.

At all times t, assume zero-coupon bonds of all maturities are available. For any given pair of time-points (t, T) with $0 \le t \le T \le T^* - h$, $f(t,T)$ denotes the forward rate on the default-free bonds applicable to the period $(T, T+h)$. The short rate is $f(t,t) = r(t)$. Forward rates follow the stochastic process:

$$f(t+h, T) \;=\; f(t,T) + \alpha(t,T)h + \sigma(t,T)X_f(t)\sqrt{h}, \tag{8}$$

where α is the drift of the process and σ the volatility; and $X_f(t)$ is a random variable taking values in the set $\{-1, +1\}$. Both α and σ are taken to be only functions of time, and not other state variables. This is done to preserve the computational tractability of the model.

We denote by $P(t,T)$ the time–t price of a default-free zero-coupon bond of maturity $T \ge t$. As usual,

$$P(t,T) \;=\; \exp\left\{ -\sum_{k=t/h}^{T/h-1} f(t,kh)\cdot h \right\} \tag{9}$$

The well-known recursive representation of the drift term α of the forward-rate and spread processes, is required to complete the risk-neutral lattice. Let $B(t)$ be the time–t value of a "money-market account" that uses an initial investment of \$1, and rolls the proceeds over at the default-free short rate:

$$B(t) = \exp\left\{ \sum_{k=0}^{t/h-1} r(kh) \cdot h \right\}. \tag{10}$$

The equivalent martingale measure Q is defined with respect to $B(t)$ as numeraire; thus, under Q all asset prices in the economy discounted by $B(t)$ will be martingales. Let $Z(t,T)$ denote the price of the default-free bond discounted using $B(t)$: $Z(t,T) = \frac{P(t,T)}{B(t)}$. which is a martingale under Q, for any $t < T$, i.e. $Z(t,T) = E_t[Z(t+h,T)]$. It follows that $Z(t+h,T)/Z(t,T) = (P(t+h,T)/P(t,T)) \cdot (B(t)/B(t+h))$. Algebraically manipulating the martingale equation leads to a recursive expression relating the risk-neutral drifts α to the volatilities σ at each t:

$$\sum_{k=t/h+1}^{T/h-1} \alpha(t,kh) = \frac{1}{h^2} \ln\left(E_t\left[\exp\left\{ -\sum_{k=t/h+1}^{T/h-1} \sigma(t,kh) X_f h^{3/2} \right\}\right]\right). \tag{11}$$

2.4 The Joint Process

We now connect the two processes for the term structure and the defaultable equity price together on a bivariate lattice. There are two goals here. First, we set up the probabilities of the joint process so as to achieve the correct correlation between equity returns and changes in the spot rate, which we denote as ρ. Second, our lattice is set up so as to be recombining, allowing for polynomial computational complexity, providing for fast computation of derivative security prices.

Specification of the joint process requires a probability measure over random shocks $[X_f(t), X_s(t)]$. This probability measure is chosen to (i) obtain the correct correlations, (ii) ensure that normalized equity prices and bond prices are martingales, and (iii) to make the lattice recombining. Our lattice model is hexanomial, i.e. from each node, there are six emanating branches or six states (of which two are absorbing states). Table 1 depicts the states.

Notice that the table contains two free parameters m_1 and m_2 in the probability measure. We solve for the correct values of m_1 and m_2 to provide a default-consistent martingale measure, with the appropriate correlation between the equity and interest rate processes, ensuring too, that the lattice recombines. The details of this derivation and the properties of the tree are presented in the Appendix. As shown there, m_1 and m_2 have the form

Table 1: Branching process and probability measure. This tableau presents the 6 branches from each node of the pricing lattice, as well as the probabilities for each branch. "def" stands for the default/absorbing state. The first four branches relate to the non-defaulted path and the last two branches lead to absorbing states.

X_f	X_s	Probability
1	1	$p_1 = \frac{1}{4}(1+m_1)[1-\lambda(t)]$
1	−1	$p_2 = \frac{1}{4}(1-m_1)[1-\lambda(t)]$
−1	1	$p_3 = \frac{1}{4}(1+m_2)[1-\lambda(t)]$
−1	−1	$p_4 = \frac{1}{4}(1-m_2)[1-\lambda(t)]$
1	def	$p_5 = \lambda(t)/2$
−1	def	$p_6 = \lambda(t)/2$

$$\begin{aligned} m_1 &= \frac{A+B}{2} \\ m_2 &= \frac{A-B}{2} \\ A &= \frac{\frac{4e^{r(t)h}}{1-\lambda(t)} - 2[a(t)+b(t)]}{a(t)-b(t)} \\ B &= \frac{2\rho}{1-\lambda(t)} \end{aligned}$$

These values may now be used Table 1. Probability bounds are presented in the Appendix in Table 4. In the special case where $\gamma = 1$, i.e. we have the basic geometric Brownian motion, and the discrete time model is implemented with the usual Cox, Ross and Rubinstein (1979) approach. In this case, the expressions above for $a(t), b(t)$ are given by $a(t) = \exp[\sigma_s\sqrt{h}]$, and $b(t) = \exp[-\sigma_s\sqrt{h}]$. However, identical results are obtained if we use the CEV model with $\gamma = 0.999999$.

2.5 The Default Process

We now account for credit risk by adding the process for default probability $[\lambda(t)]$ to the lattice. One way to do this is to embed a separate $\lambda(t)$ process, but this increases implementational complexity by adding an extra dimension to the lattice model. To facilitate easy implementation, we we make default a function of equity prices and interest rates at each node. There are good theoretical reasons also for doing this. First, equity prices already reflect credit risk, so there is a connection between $\lambda(t)$ and equity prices. Second, default probabilities have been shown to be connected to the term structure (see Duffie, Saita and Wang (2005) who find that default probabilities are functions of the equity index and term structure).

Our approach specifies a *conditional* $\lambda(t)$ at each node, i.e. rather than add a separate default probability process, we simply make the $\lambda(t)$s a function of the state variables of equity and interest rates. We refer to this as an *endogenous* default approach.[5] If in fact, default probabilities were added as a separate stochastic process (which we denote the *exogenous* approach, as in Davis and Lischka (1999) or Andersen and Buffum (2002)), the question of consistency conditions between $\lambda(t)$, equity and interest rates would arise, a complex situation to resolve. By positing a functional relationship of $\lambda(t)$ to the other variables, we are able to obtain a consistent lattice as well as a more parsimonious one. As noted before, $\lambda(t) = 1 - e^{-\xi(t)h}$, and we express the default intensity $\xi(t)$ as:

$$\xi[\mathbf{f}(t), S(t), t; \theta] \in [0, \infty) \tag{12}$$

i.e., a function of the term structure of forward rates $\mathbf{f}(t)$, the stock price $S(t)$ at each node, and time t. This function may be as general as possible. We impose the condition that is required of default intensities, i.e. $\xi(t) \geq 0$. θ is a parameter set that defines the function. This is not a new approach. A similar *endogenous* default intensity extraction has been implemented in Das and Sundaram (2000), Carayannopoulos and Kalimipalli (2003), and Acharya, Das and Sundaram (2002). However, the settings in those papers were less general than in this one.

Of course, in addition to the probability of default of the issuer, a recovery rate is required in the two states in which default occurs. The recovery rate may be treated as constant, or as a function of the state variables in this model. It may also be pragmatic to express recovery as a function of the default intensity, supported by the empirical analysis of Altman, Brady, Resti and Sironi (2002).

Various possible parameterizations of the default intensity function may be used. For example, the following model (subsuming the parameterization of Carayannopoulos and Kalimipalli (2003)) prescribes the relationship of the default intensity $\xi(t)$ to the stock price $S(t)$, short rate $r(t)$, and time on the lattice $(t - t_0)$.

$$\begin{aligned} \xi(t) &= \exp[a_0 + a_1 r(t) - a_2 \ln S(t) + a_3(t - t_0)] \\ &= \frac{\exp[a_0 + a_1 r(t) + a_3(t - t_0)]}{S(t)^{a_2}} \end{aligned} \tag{13}$$

For $a_2 \geq 0$, we get that as $S(t) \to 0$, $\xi(t) \to \infty$, and as $S(t) \to \infty$, $\xi(t) \to 0$.

2.6 Example: two-period tree

Here, we present a simple illustration of a pricing tree in 3 dimensions, one each for time (t), interest rate (i), and stock price (j) - a tree in (t, i, j) space. The initial node is denoted $(1, 1, 1)$ and after one period, we have 4 nodes (interest rates go up and down, and stock price goes up and down), denoted $\{(2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$. Since the tree is recombining, after 2 periods, we will

[5] Wall Street practitioners who attended presentations of the paper called this a "two and a half" dimensional model since the third dimension (default) is determined partly as a function of the first two (equity and interest rates).

Table 2: Two-Period Tree Example. In this table, we present the results of a two-period tree based on given input parameters. The example here may be useful for anyone replicating our model to check their results. The input parameters are the default function values $\{a_0, a_1, a_2, a_3\}$, the stock price S, stock volatility σ_s, correlation of term structure with stock prices ρ, and the time step on the tree h. The initial forward rate term structure and corresponding volatilities are also given. The output price lattice is recombining, and therefore, there are $(n+1)^2$ nodes at the end of the n^{th} period on the lattice. The lattice starts at node $(1,1,1)$ and then moves to 4 nodes in the subsequent period, and then to 9 nodes, etc. At each time step there are two axes (i,j) for interest rates and stock prices respectively. The default probability (λ) for the next period is also stated at each node, and is a function of r, S, and time. The default function is: $\lambda = 1 - \exp[-\xi h]$, where $\xi = \exp[a_0 + a_1 r + a_3 ih]/(S^{a_2})$, where i indexes nodes on the interest rate branch of the tree (see table below). Note that a_3 modulates the slope more severely when rates are low than high. Alternative specifications would be to replace i with t. It can be seen that the default probability declines as S increases, and increases in r.

Input Values

Parameter	Value	Fwd Rates	FWR Vols
a_0	0.1	0.060	0.0020
a_1	0.1	0.065	0.0019
a_2	1.0	0.070	0.0018
a_3	0.1		
S	100		
σ_s	0.40		
ρ	0.4		
h	0.5		
γ	1		

Output Price Lattice

[t	i	j]	r	S	λ
1	1	1	0.0600	100.0000	0.0058
2	1	1	0.0663	132.6896	0.0044
2	1	2	0.0663	75.3638	0.0077
2	2	1	0.0637	132.6896	0.0046
2	2	2	0.0637	75.3638	0.0081
3	1	1	0.0725	176.0654	0.0033
3	1	2	0.0725	100.0000	0.0058
3	1	3	0.0725	56.7971	0.0102
3	2	1	0.0700	176.0654	0.0035
3	2	2	0.0700	100.0000	0.0061
3	2	3	0.0700	56.7971	0.0108
3	3	1	0.0675	176.0654	0.0037
3	3	2	0.0675	100.0000	0.0064
3	3	3	0.0675	56.7971	0.0113

have only 9 nodes. In the illustration below, presented in Table 2, are the results of calculations for 2 periods. At each node, we show the one-period probability of default. The table presents all the details of the inputs used in the example. This table should be useful to readers who wish to implement the model, and it also details all the inputs required for building the pricing lattice. We have assumed the CRR process for the equity model, i.e. the CEV coefficient is $\gamma = 1$. As may be seen, the approach requires a parsimonious set of inputs, all of which are observable and may be accessed from standard sources.

3 Implementation

We open with a discussion of how credit default swaps may be used to help calibrate the model. Then, we present examples showing how different parameters result in various default swap spread term structures. Following this, we look at how default risk results in differences in prices of corporate bonds, with or without convertible features. Finally, we present a section explaining how the model may be used for pricing credit correlation products.

3.1 Calibrating the model with credit default swaps

A default swap is a contract between two parties, whereby the buyer of the default swap pays a flat stream of insurance payments to the seller, who makes good any loss on default of a reference credit (hereafter, "bond"). The seller's payment is contingent upon default. The price of a default swap is quoted as a spread rate per annum. Therefore, if the default swap rate is 100 bps, paid quarterly, then the buyer of the insurance in the default swap would pay 25 bps of the notional each quarter to the seller of insurance in the default swap. The present value of all these payments must equal the expected loss on default anticipated over the life of the default swap. In the event of default, the buyer of protection in the default swap receives the par value of the bond less the recovery on the bond. In many cases, this is implemented by selling the bond back to the insurance seller at par value. Pricing of a default swap has been described in detail in Duffie (1999).

The increasing amount of trading in default swaps now offers a source of empirical data for calibrating the model. Other models, such as CreditGrades[6], also use default swap data. Hence, the term-structure of default swaps is now available for cross-sectional fitting of our model parameters. A recent paper by Longstaff, Mithal and Neis (2005) undertakes an empirical comparison of default swap and bond premia in a parsimonious closed-form model.

The way the lattice is set up in our model makes it very simple to compute the default swap spread s (stated as a rate in basis points per annum). Since the probability of default is known at each node on the tree, we can compute the expected cashflow from the default swap at each node, which is just $[\lambda(t) \times (\text{Loss in Value on Default})]$. We accumulate these values at each node and discount them back along the tree to obtain the expected present value of loss payments made by the writer of the default swap. The buyer then pays in a constant spread s each period, such that the present value of these payments equals the present value of expected loss on default.

Assume that we have "pure" default swap spreads for a range of maturities, $t = 1, 2, 3, \ldots, T$ years. The pure premium on a default swap is the present value of insurance payments on a defaultable zero-coupon bond. The premium is equal to the expected present value of payouts on default of the underlying zero-coupon instrument. Expectations are taken under the default-risk adjusted martingale measure described in this paper. Given any four maturities, we can calibrate the four parameters $\theta = \{a_0, a_1, a_2, a_3\}$ in the function in equations (12) and (13) by exact fitting of four default swap premia. If more than four maturities for default swap spreads are available, the parameters may be fitted using a least squares criterion.

We denote the recovery rate on default as ϕ, which we take to be a constant. Applying the recovery of market value (RMV) assumption on default, the pure default swap rate is the continuous stream of payments expressed in basis points that equates the present value of these payments to the expected present value of the payoffs on the default swap. On the lattice, these values may be

[6]This is a model developed by RiskMetrics, which uses default swaps to calibrate an extended Merton-type model to obtain probabilities of default.

computed via backward recursion.

Since we are working in discrete time, we have to be careful about modeling the timing of default. We make the following assumptions. (a) In any period in which default occurs, recovery payoffs are realized at the end of the period. (b) Default is based on the default intensity at the beginning of the period. In order to price a credit default swap we define the following quantities.

First, we define the price of a defaultable zero-coupon bond. We denote the price of this bond at time t as $ZCCB(t)$. The pricing recursion under the RMV condition is as follows:

$$\begin{aligned} ZCCB(t) &= e^{-r(t)h} \left\{ \sum_{k=1}^{4} \hat{p}_k(t) ZCCB_k(t+h) \right\} [1 - \lambda(t)(1-\phi)], \\ & \quad ZCCB(T) = 1.0 \end{aligned} \tag{14}$$

Here, $\hat{p}_k(t) = p_k(t)/[1-\lambda(t)], k = 1..4$ are the four probabilities for the non-default branches of the lattice, *conditional* on no default occurring, and k indexes the four states of non-default. Therefore, $\sum_{k=1}^{4} \hat{p}_k(t) = 1, \forall t$. We need to price this at every node on the lattice, as we will define recovery as a fraction of the value of this bond, if and when it occurs.

Second, we compute the expected present value of all payments in the event of default of the zero-coupon bond, denoted $CDS(t)$. Again, the lattice-based recursive expression is:

$$\begin{aligned} CDS(t) &= e^{-r(t)h} \left\{ \sum_{k=1}^{4} \hat{p}_k(t) CDS_k(t+h) \right\} [1 - \lambda(t)] \\ &+ \lambda(t) ZCCB(t)(1-\phi), \quad CDS(T) = 0.0. \end{aligned} \tag{15}$$

The formula above has two components. *(i)* The first part is the present value of future possible losses on the default swap, given that default does not occur at time t. *(ii)* The second part is the present value of the loss (sustained at the end of the period). Note that the formula contains $ZCCB(t)(1-\phi)$, which is the present value of loss at the end of the period, $ZCCB(t+h)(1-\phi)$.

Third, we calculate the expected present value of a \$1 payment at each point in time conditional on no default occurring. This is defined as follows:

$$G(t) = \left[e^{-r(t)h} \left\{ \sum_{k=1}^{4} \hat{p}_k(t) G_k(t+h) + 1 \right\} \right] [1 - \lambda(t)], \;\; G(T) = 0.0. \tag{16}$$

This is computed since we only make default swap premium payments until the period in which the default occurs. Therefore, we wish to compute and store the value of unit \$1 payments each period provided default has not occured. It is assumed that the payments are made at the end of the period conditional on no default in that period.

In order to get the annualized basis points spread (s) for the premium payments on the default swap, we equate the quantities $s \times h \times G(0) = CDS(0)$, and the premium spread is:

$$s = \frac{CDS(0)}{h \times G(0)} \times 10,000 \text{ bps.} \tag{17}$$

In the equation above, we multiply by 10,000 and divide by the time interval h in order to convert the amount into annualized basis points. We use this calculation in the illustrative examples that are provided in the following section.

3.2 Default Swap Spread Curves

In this section we demonstrate that the model is able to generate varied spread curve shapes. In the plots in Figure 1 we present the term structure of default swap spreads for maturities from 1 to 5 years. The default intensity is specified as $\xi(t) = \exp[a_0 + a_1 r(t) + a_3(t - t_0)]/S(t)^{a_2}$. Keeping a_0 fixed, we varied parameters a_1 (impact of the short rate), a_2 (impact of the equity price) and a_3 (impact of the term structure of credit premia) over two values each. Four plots are the result. The other inputs to the model, such as the forward rates and volatilities, stock price and volatility, etc., are provided in the description of the figure. Comparison of the plots provides an understanding of the impact of the parameters.

When $a_3 > 0$, the term structure of default swap spreads is upward sloping, as would be expected. When $a_3 < 0$, i.e. default spreads are declining, consistent with a reduction in premia over time. Hence, we may think of a_3 as the slope parameter in the model. Comparison of the plots also shows the effect of parameter a_2, the coefficient of the equity price $S(t)$. As $a_2 > 0$ increases, default spreads decline as the stock price lies in the denominator of the default intensity function, as can be seen in the plots. A comparison of curves in Figure 1 shows that parameter a_1, the coefficient on interest rates, has a level effect on the spread curve. In sum, our four parameter default function is flexible enough to capture a variety of economic phenomena, as well as generate a spectrum of curve shapes.

3.3 The Impact of Leverage

Because our model is based on a default-extended CEV process, varying the CEV coefficient γ enables the simulation of varied leverage effects. If we assume the CRR model ($\gamma = 1$) as a base case, then reductions in γ will increase the leverage effect. Figures 2 and 3 show how changes in γ impact the term structure of CDs spreads. The diffusion coefficient in the CEV model is set up so that the total volatility is roughly the same across varied choices of γ so as to result in a meaningful comparison. For this, we follow Nelson and Ramaswamy (1990) in setting σ_s to satisfy the following condition: $\sigma_s S(0)^\gamma = \sigma_{CRR} S(0)$ (total CEV volatility at inception equals total CRR volatility). From all three figures, we see that increases in the leverage effect result in an increase in spreads indicating that the direction of the impact conforms to theory. However, the change in leverage appears to have only a small quantitative effect on spreads, suggesting that the level of total volatility matters more in the pricing of CDS than the particular form of the volatility function. For simplicity, therefore, in the remainder of the paper, we set $\gamma = 1$ (i.e, use the defaultable CRR model) in examples and computations.

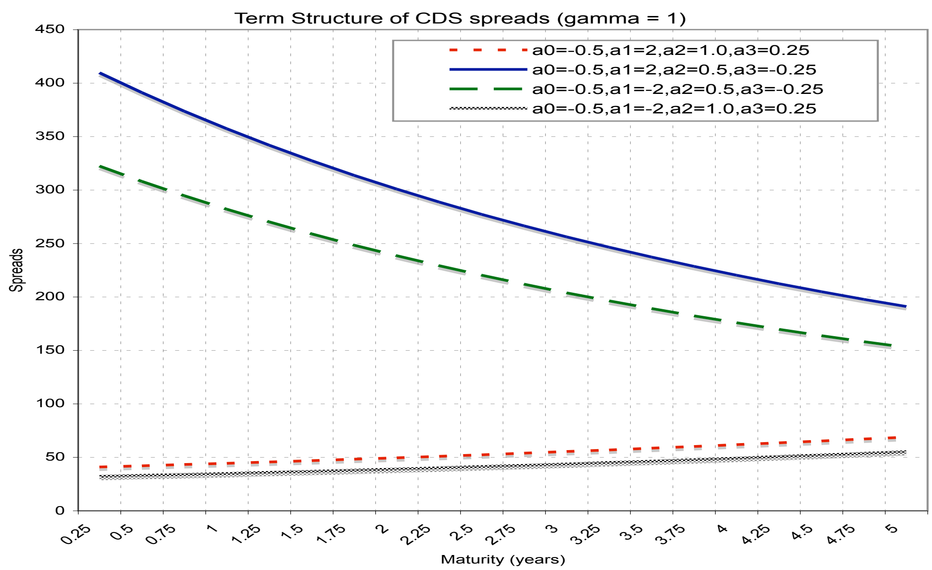

Figure 1: Term structure of default swap spreads for varied default function parameters. This figure presents the term structure of default swap spreads for maturities from 1 to 5 years. The figure has 4 plots. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a_3(t - t_0)]/S(t)^{a_2}$. Keeping all the other parameters fixed, we varied parameters a_1, a_2 and a_3. Hence, the 4 plots are the result. Periods in the model are quarterly, indexed by i. The forward rate curve is very simple and is just $f(i) = 0.06 + 0.001\ln(i)$. The forward rate volatility curve is $\sigma_f(i) = 0.01 + 0.0005\ln(i)$. The initial stock price is 100, and the stock return volatility is 0.30, given $\gamma = 1$. Correlation between stock returns and forward rates is 0.30, and recovery rates are a constant 40%. The default function parameters are presented on the plots.

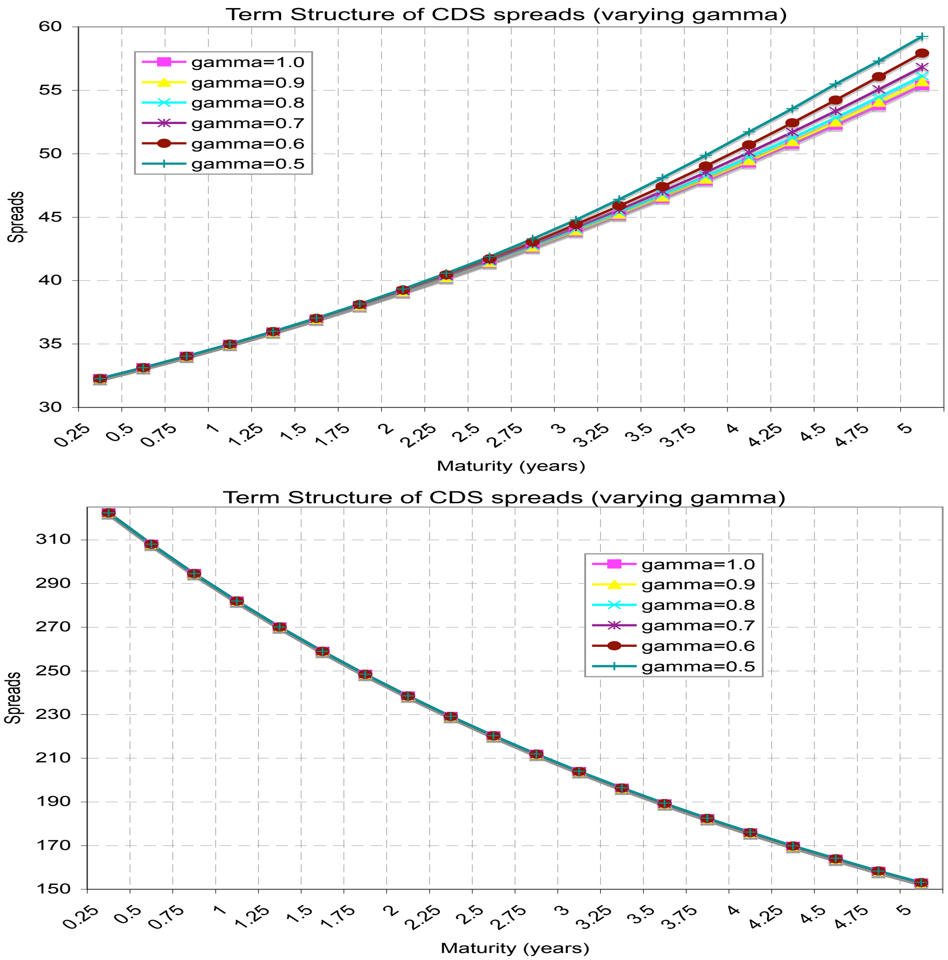

Figure 2: Term structure of default swap spreads for varying leverage. This figure presents the term structure of default swap spreads for maturities from 1 to 5 years. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a_3(t - t_0)]/S(t)^{a_2}$. The figure has 2 graphs. Keeping all the other parameters fixed, we fixed parameters $a_0 = -0.5$, $a_1 = -2$,$a_2 = 1$ and $a_3 = 0.25$ for the first plot and $a_0 = -0.5$, $a_1 = -2$,$a_2 = 0.5$ and $a_3 = -0.25$ for the second. Periods in the model are quarterly, indexed by i. The forward rate curve is very simple and is just $f(i) = 0.06 + 0.001 \ln(i)$. The forward rate volatility curve is $\sigma_f(i) = 0.01 + 0.0005 \ln(i)$. The initial stock price is 100, and the stock return volatility is 0.30, given $\gamma = 1$. Since the stock volatility when $\gamma = 1$ is 0.4, as we change the CEV coefficient γ, we also adjust the variable σ so as to keep the conditional total volatility of the diffusion roughly the same, by the following equation: $\sigma_s S_0^\gamma = \sigma_{CRR} S_0$. This ensures that the total diffusion volatility in the CEV model is approximately the same as in the CRR model, and is the same approach as used by Nelson and Ramaswamy (1990) for comparisons in their paper. Correlation between stock returns and forward rates is 0.30, and recovery rates are a constant 40%. We varied γ from 0.5 to 1.0. We notice very minor changes in CDS spread curves.

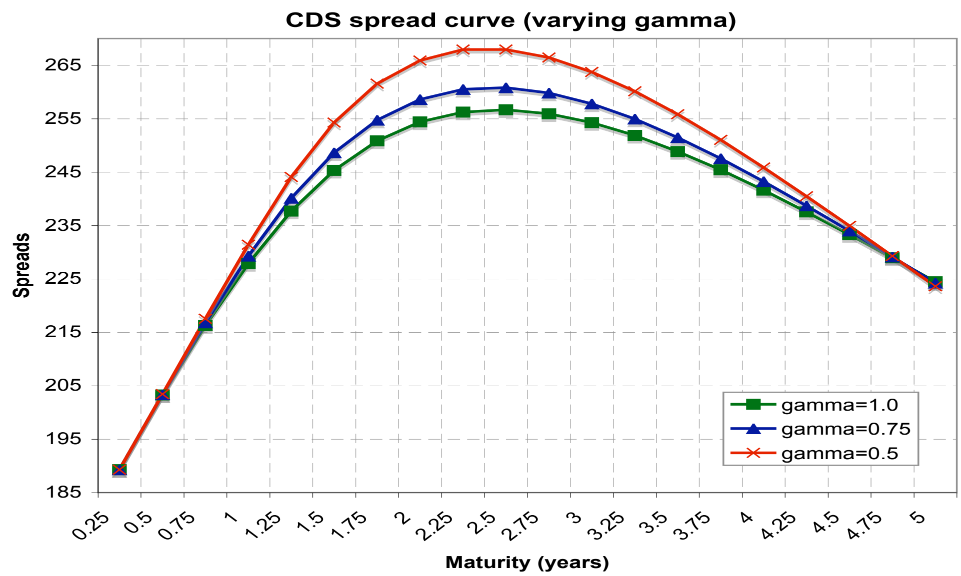

Figure 3: Term structure of default swap spreads for varying leverage. This figure presents the term structure of default swap spreads for maturities from 1 to 5 years. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a_3(t - t_0)]/S(t)^{a_2}$. Keeping all the other parameters fixed, we fixed parameters $a_0 = -1.814681$, $a_1 = 58.55167$,$a_2 = 2.16029$ and $a_3 = -0.37119$. Periods in the model are quarterly, indexed by i. The forward rate curve is very simple and is just $f(i) = 0.06 + 0.001\ln(i)$. The forward rate volatility curve is $\sigma_f(i) = 0.01 + 0.0005\ln(i)$. The initial stock price is 58.31, and the stock return volatility is 0.40, given $\gamma = 1$. Correlation between stock returns and forward rates is 0.0, and recovery rates are a constant 40%. We varied γ from 0.5 to 1.0. Since the stock volatility when $\gamma = 1$ is 0.4, as we change the CEV coefficient γ, we also adjust the variable σ so as to keep the conditional total volatility of the diffusion roughly the same, by the following equation: $\sigma_s S_0^\gamma = \sigma_{CRR} S_0$. This ensures that the total diffusion volatility in the CEV model is approximately the same as in the CRR model, and is the same approach as used by Nelson and Ramaswamy (1990) for comparisons in their paper. We notice very minor changes in CDS spread curves.

3.4 Impact of default risk on embedded options

The model may be easily used to price callable-convertible debt. One aspect of considerable interest is the extent to which default risk impacts the pricing of convertible debt, through its effect on the values of the call feature (related to interest rate risk) and the convertible feature (related to equity price risk). We chose an initial set of parameters to price convertible debt, and examined to what extent changing levels of default risk impacted a plain vanilla bond versus a convertible bond. The parameters and results are presented in Figure 4.

Given the base set of parameters, we varied a_0 from 0 to 4. As a_0 increases, the level of default risk increases too. For each increasing level of default risk, we plot the prices of a defaultable plain vanilla coupon bond with no call or convertible features. We also plot the prices of a callable-convertible bond. Note that this numerical experiment has been kept simple by setting $a_1 = a_3 = 0$, so that there are no interest-rate and term effects on the default probabilities.

The results comparing the plain coupon bond with a callable-convertible coupon bond are presented in Figure 4 (upper panel). The value of a_0 is varied from 0 (low default risk) to 4 (higher risk). Bond values decline as default risk (a_0) increases. As default risk increases, the difference in price between the callable-convertible and vanilla bonds declines rapidly and eventually goes to zero. Since default risk effectively shortens the duration of the bonds, it also reduces the value of the call option. Hence, the price difference between the vanilla bond and the callable-convertible bond declines as a_0 increases. Moving from the upper to middle panel is based on one change, i.e. equity volatility was increased from 20% per year to 40% per year. The results are the same, but bond prices converge faster. Hence at high equity volatility, default risk impacts the convertible value faster, as there are more regions in the state space on our pricing tree with greater probability of default. Therefore, default risk systematically impacts the commingled values of interest rate calls and equity convertible features in debt contracts. By shortening the effective duration of the bond, both options decline in value, driving the price of the callable-convertible closer to that of the vanilla bond (see Buchan (1998) for early work on such effects in the pricing of convertible bonds).

In Figure 5 we vary the dependence of default risk on equity prices. The base case is presented in the upper panel of the figure when the coefficient $a_2 = 1$. In the lower panel, we changed $a_2 = 0.75$, resulting in higher default risk. Hence, the prices are lower in the lower panel. The values of parameters for the conversion feature and for the call feature were chosen so as to make the plain bond and the callable-convertible equal in price in the upper panel. Reducing the value of a_2 to inject more default risk in fact increases the price of the callable-convertible relative to that of the plain bond, and the effect is higher for greater levels of default risk. Here, increases in default risk tend to increase the difference between equity call option values and the bond callable feature, ceteris paribus, and this drives an increasing wedge between the convertible bond and the plain bond. Further, the level of parameter a_2 also determines whether defaultable bond prices are convex or concave in default risk, and both possibilities are pictured in the two panels of Figure 5. At lower levels of default risk, the convertible bond is concave in a_0, and at higher levels it becomes convex.

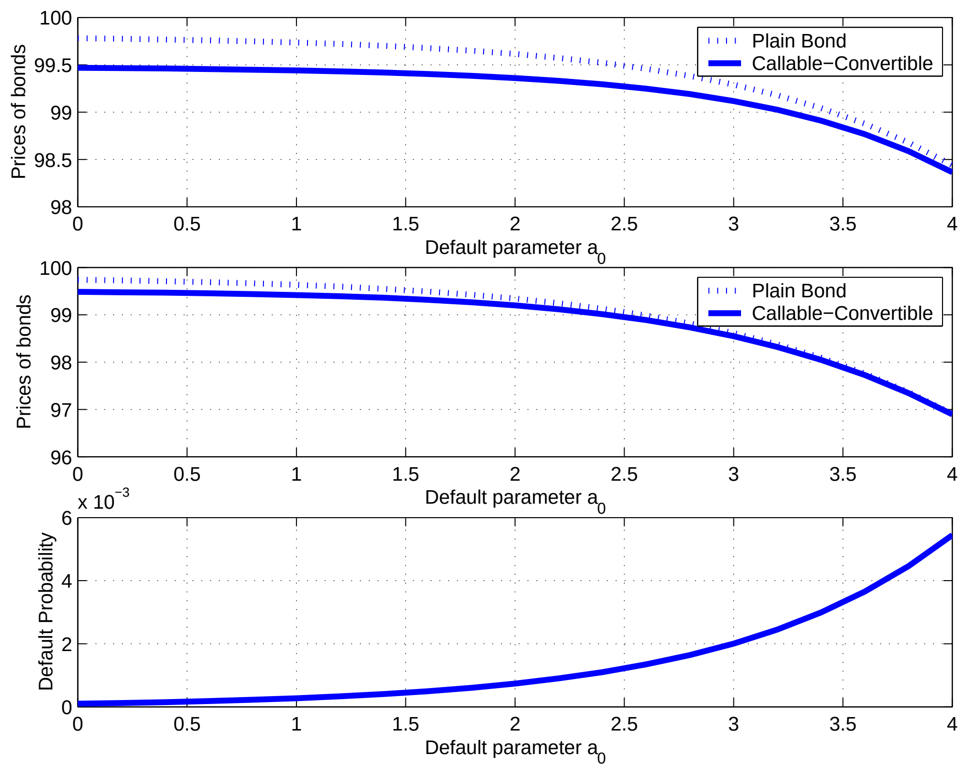

Figure 4: Comparison of callable convertible bonds and plain defaultable bonds in different volatility environments. We assumed a flat forward curve of 6%. We also assumed a flat curve for forward rate volatility of 20 basis points per period. The maturity of the bonds is taken to be 5 years, and interest is assumed paid quarterly on the bonds at an annualized rate of 6%. Default risk is based on default intensities which come from the model in equation (13). The base parameters for this function are chosen to be $a_0 = 0$, $a_1 = 0$, $a_2 = 2$, and $a_3 = 0$. Under these base parameters default risk varies only with the equity price. In our numerical experiments we will vary a_0 to examine the effect of increasing default risk. The stock price is $S(0) = 100$. The recovery rate on default is 0.4, and the correlation between the stock return and term structure is 0.25. If the bond is callable, the strike price is 100. Conversion occurs at a rate of 0.3 shares for each bond. The dilution rate on conversion is assumed to be 0.75. This figure contains three panels. The upper panel presents a comparison of bond prices when equity volatility is set to 20% (for $\gamma = 1$), and the default probability parameter a_0 is varied on the x-axis. The middle panel shows the same comparison when the volatility is 40%. The bottom panel shows the corresponding default probability.

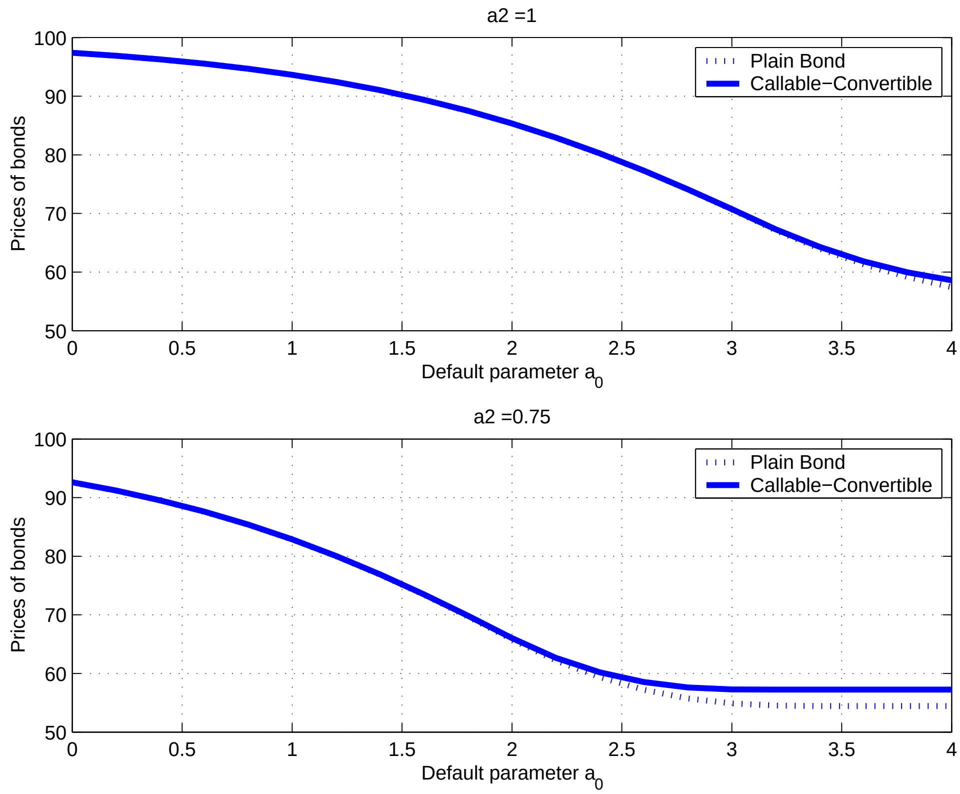

Figure 5: Comparison of default risk effects on callable convertible bonds and plain defaultable bonds for different equity dependence. We assumed a flat forward curve of 6%. We also assumed a flat curve for forward rate volatility of 20 basis points per period. The maturity of the bonds is taken to be 5 years, and interest is assumed paid quarterly on the bonds at an annualized rate of 6%. Default risk is based on default intensities which come from the model in equation (13). The base parameters for this function are chosen to be $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, and $a_3 = 0$. Under these base parameters default risk varies only with the equity price. In our numerical experiments we will vary a_0 to examine the effect of increasing default risk. The stock price is $S(0) = 100$ and stock volatility is 20% (for $\gamma = 1$). The recovery rate on default is 0.4, and the correlation between the stock return and term structure is 0.25. If the bond is callable, the strike price is 105. Conversion occurs at a rate of 0.3 shares for each bond. The dilution rate on conversion is assumed to be 0.75. This figure contains two panels. The upper panel presents a comparison of bond prices when $a_2 = 1$, and the default probability parameter a_0 is varied on the x-axis. The lower panel shows the same comparison when $a_2 = 0.75$, which is higher default risk.

Therefore, depending on market conditions, and the level of default risk, increases in default risk may increase or decrease the price differential of two bonds that have embedded options. This highlights the need for careful consideration of default risk effects using an appropriate model that considers all forms of risk and their interactions.

3.5 Correlated default analysis

The model may be used to price credit baskets. There are many flavors of these securities, and some popular examples are n^{th} to default options, and collateralized debt obligations (CDOs). These securities may be valued using Monte Carlo simulation, under the risk-neutral measure, based on the default functions fitted using the techniques developed in this paper.

The first step in modeling default correlations is to model the correlation of default intensity amongst issuers. Since our model calibrates default functions $\xi(\mathbf{f}(t), S(t), t)$ for each issuer, credit correlations are determined from the correlations of the forward curve, and issuer stock prices, which are observable. Suppose we are given the function for default intensity of issuer $i, i = 1...n$, as $\xi_i(t) = \exp[a_0^i + a_1^i r(t) + a_3^i (t - t_0)]/S_i(t)^{a_2^i}$. Let the covariance matrix of $[r(t), S_1(t), S_2(t), ..., S_n(t)]'$ be Σ. Then, the covariance matrix of default intensities $\{\xi_i(t)\}_{i=1...n}$ is $V(t) \approx J(t)\Sigma J(t)'$, where $J(t) \in R^{n\times(n+1)}$ is the Jacobian matrix whose i^{th} row is as follows:

$$J_i(t) = \left[\frac{\partial \xi_i(t)}{\partial r(t)}, 0, .., 0, \frac{\partial \xi_i(t)}{\partial S_i(t)}, 0, ..., 0\right] = \left[a_1^i \xi_i(t), 0, .., 0, \frac{-a_2^i}{S_i(t)} \xi_i(t), 0, .., 0\right]$$

We may contrast this approach with the somewhat ad-hoc practice of using equity correlations as a proxy for asset correlations, used in turn to drive default correlations in structural models. Our method is closer to the approach, also used in practice, of a factor structure that drives default correlations. However, our approach has a significant advantage over other factor models – i.e. we calibrate each default function in a manner that is based on observables, *and* is also consistent with a no-arbitrage model over default, equity and interest-rate risks. A comprehensive examination of credit correlations in this framework is undertaken in Bandreddi, Das and Fan (2005).[7]

4 Default Risk Premia

In this section, we present a simple empirical application of the model by fitting it to the 30 names in the CDX INDU Index. Our calibration will permit us to extract credit risk premia.

The model is calibrated as follows. The stock price is taken from `CreditGrades`. Stock volatility is based on a historical estimate from 1000 days past returns. To obtain the forward curve of interest rates each day, we extracted constant maturity yields for all maturities upto five years available from the Federal Reserve web site[8], and through the standard bootstrapping approach, converted the

[7] This approach to credit correlations is a "bottom-up" model, similar to the popular class of copula models. In contrast are the "top-down" class of models, for example, Giesecke and Goldberg (2005) and Longstaff and Rajan (2006).

[8] `http://www.federalreserve.gov/releases/h15/data.htm`

yields into forward rates at quarterly intervals. Where necessary, linear interpolation is used. The interest rate volatility is computed as the historical volatility of each forward rate over the sample period. The correlation between equity and interest rates was set equal to the historical correlation between the stock return and the 3-month interest rate, computed on a rolling basis, with one-year histories. The CDS spreads for maturities from 1-5 years are taken from `CreditGrades`. The four parameters of the default function are fitted to these CDS spreads using `Matlab`. A least squares difference of the CDS spreads to model spreads is undertaken to obtain best fit. Once calibrated, the one-year risk-neutral probability of default (PD) is calculated (described next).

We compare the default probabilities from `CreditGrades` (that are under the physical measure, and represent the real world probabilities of default), to the risk-neutral probabilities that we extract from the CDS spreads. The ratio of the risk-neutral probabilities to the real world ones (usually greater than one), are a metric of the risk premium in the market for credit risk. See Berndt, Douglas, Duffie, Ferguson and Schranz (2005) for a comprehensive look at default risk premia extraction from CDS and expected default frequencies. For our analysis we use the one-year default probabilities.

Since our model develops a function for default intensities $\xi_i(t)$, for each issuer i, the one year probability of default is a function of the expected integrated intensity for one year, taken under the risk-neutral measure, which is $Y = E^* \left[\int_0^1 \xi_i(t) dt\right]$, where E^* is the expectations operator under the risk-neutral measure. We undertook this integration using a tree. Given the integral, the one-year probability of default is $[1 - \exp(-Y)]$.

Empirical Analysis

Using data from `CreditGrades` for the period from January 2000 to June 2002, for the 30 issuers from the CDX INDU Index, we calibrated the model each day to CDS spreads. Using the fitted parameters, we computed the one-year risk-neutral default probabilities, and quantified the risk premia by dividing the risk-neutral default probability by the default probability from `CreditGrades`. Figure 6 shows the plot of the average default probabilities (equally weighted across 30 firms) over time, and Table 3 shows the average premia over time. This corresponds to the measure presented in Berndt, Douglas, Duffie, Ferguson and Schranz (2005).

A principal components decomposition of risk premia shows that there are two main components, as shown in Figure 7. We also compared the time series of the main principal component to the time series of the S&P500 index, and found them to track closely, with a correlation of 50%. See Figure 8. This finding has connections to Duffie, Saita and Wang (2005) where the S&P index is found to contain predictive value for defaults. We also compared the principal components to the time series of the VIX (volatility) index. In this case, the correlations were not significantly different from zero.

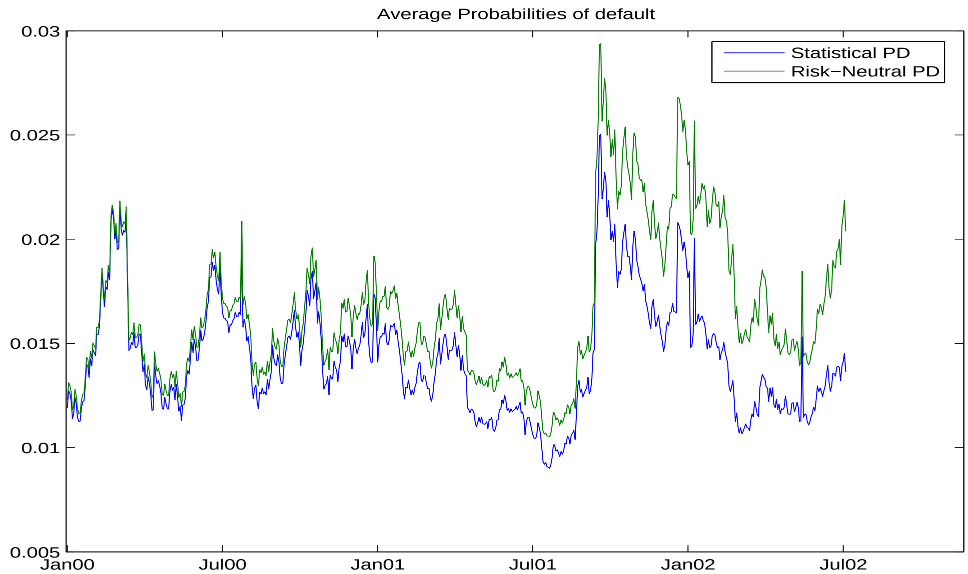

Figure 6: Average risk-neutral 1-year probabilities of default plotted against those under the statistical measure. The data comes from 30 firms in the Dow Jones CDX index. The period covered is January 2000 to June 2002.

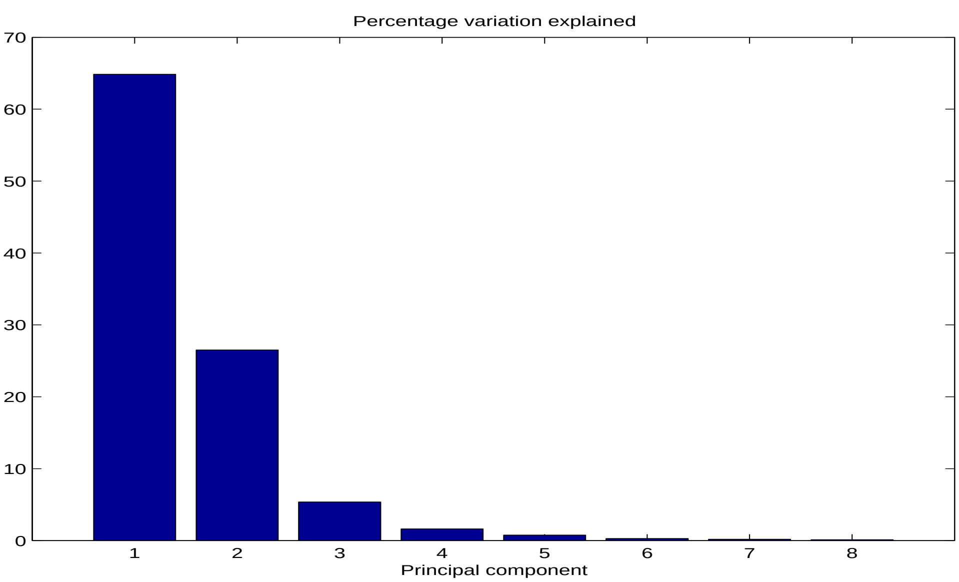

Figure 7: Principal components decomposition of average credit risk premia, computed as the ratio of risk-neutral 1-year probabilities of default to those under the statistical measure. The data comes from 30 firms in the Dow Jones CDX index. The period covered is January 2000 to June 2002. There are 2 main components.

Table 3: Issuers with respective risk premium ratios and date ranges. The dates are in the form YYYY-MM-DD. The issuers are primarily from the Dow Jones CDX index set. The risk premium is the average ratio of the risk-neutral default probability to that under the physical measure.

Ticker	No. of Obs.	Start Dt	End Dt	Risk Premium
AL	531	2000-04-28	2002-07-05	3.6935
AA	608	2000-01-03	2002-07-05	7.4249
CSX	608	2000-01-03	2002-07-05	1.0750
DE	608	2000-01-03	2002-07-05	1.0955
F	608	2000-01-03	2002-07-05	1.0463
BA	608	2000-01-03	2002-07-05	2.2341
CAT	608	2000-01-03	2002-07-05	1.1370
CTX	607	2000-01-04	2002-07-05	1.1144
GR	608	2000-01-03	2002-07-05	1.9886
IP	608	2000-01-03	2002-07-05	1.4127
DOW	608	2000-01-03	2002-07-05	3.4821
NSC	608	2000-01-03	2002-07-05	1.1005
LEN	608	2000-01-03	2002-07-05	1.3695
RTN	196	2001-09-17	2002-07-05	1.3161
TXT	608	2000-01-03	2002-07-05	1.1899
UNP	608	2000-01-03	2002-07-05	1.2366
ROH	608	2000-01-03	2002-07-05	1.9879
WY	608	2000-01-03	2002-07-05	1.6437
PHM	608	2000-01-03	2002-07-05	1.1663
MWV	608	2000-01-03	2002-07-05	2.0782
EMN	608	2000-01-03	2002-07-05	1.2730
NOC	608	2000-01-03	2002-07-05	1.4382
BNI	608	2000-01-03	2002-07-05	1.1417
LMT	608	2000-01-03	2002-07-05	1.4602
LEA	608	2000-01-03	2002-07-05	1.0968
HOM	608	2000-01-03	2002-07-05	9.0993
AXL	369	2000-12-29	2002-07-05	1.1679
GM	608	2000-01-03	2002-07-05	1.2862
IR	608	2000-01-03	2002-07-05	1.6958
DD	606	2000-01-03	2002-07-05	14.7494

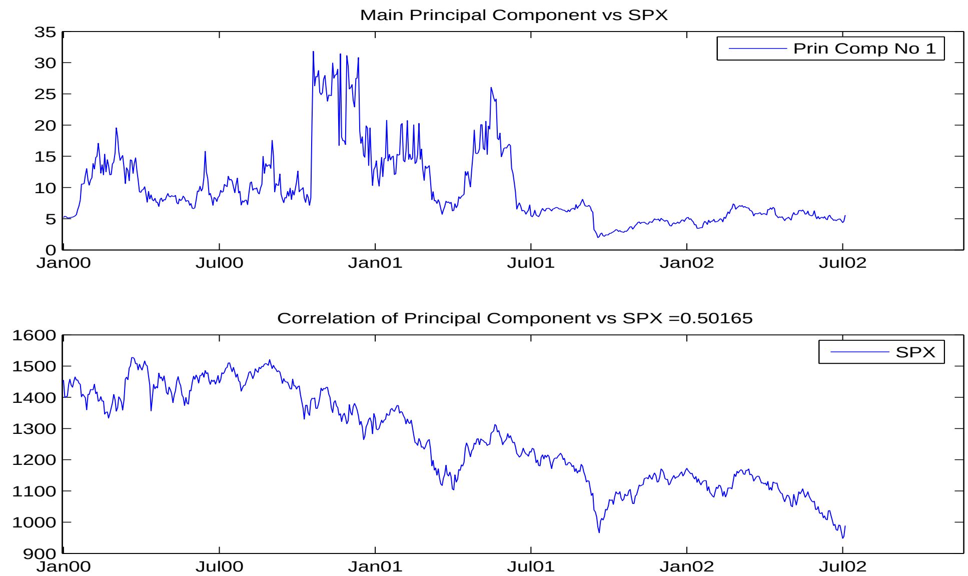

Figure 8: Comparison of the first principal component of credit risk premia against the S&P500 index. The main component comes from a principal components decomposition of average credit risk premia, computed as the ratio of risk-neutral 1-year probabilities of default to those under the statistical measure. The data comes from 30 firms in the Dow Jones CDX index. The period covered is January 2000 to June 2002. There are 2 main components. The correlation between the main component and the S&P500 index is 50%.

5 Concluding Comments

The following economic objectives are met by our model. One, we develop a pricing model covering multiple risks, which enables security pricing for hybrid derivatives with default risk. Two, the model enables the extraction of easy to calibrate default probability functions for state-dependent default. Three, using observable market inputs from equity and bond markets, we value complex securities via relative pricing in a no-arbitrage framework, e.g.: debt-equity swaps, distressed convertibles. Four, the model is useful in managing credit portfolios and baskets, e.g. collateralized debt obligations (CDOs) and basket default swaps. Finally, the extraction of credit risk premia is feasible in the model.

Technically, our hybrid defaultable model combines the ideas of both, structural and reduced-form approaches. It is based on a risk-neutral setting in which the joint process of interest rates and equity are modeled together with the boundary conditions for security payoffs, after accounting for default. We use a default-extended CEV equity model that allows volatility to vary in accordance with the leverage effect from structural models. The martingale measure in the paper is default consistent. The model is embedded on a recombining lattice, providing fast computation with polynomial complexity for run times. Cross-sectional spread data permits calibration of an implied default probability function which dynamically changes on the state space defined by the pricing lattice. The model is parsimonious, and we have been able to implement it on a spreadsheet. Further research, directed at parallelizing the algorithms in this paper, and improving computational efficiency is predicated and under way. On the economic front, the model's efficacy augurs well for extensive empirical work.

Appendix

A Deriving the branching process with risk-neutral probabilities

In order for the normalized equity process to be a martingale, we require that at every node, we adjust the probability measure over $X_s(t)$ such that

$$E\left[\frac{S(t+h)}{S(t)}\right] = \exp[r(t)h].$$

In addition, under the HJM model, the mean value of the random variable X_f must be zero, and its variance should be 1. These properties are verified as follows.

$$\begin{aligned} E(X_f) &= \frac{1}{4}[1+m_1+1-m_1-1-m_2-1+m_2](1-\lambda(t)) \\ &\quad +\frac{\lambda(t)}{2}[1-1] \\ &= 0 \\ Var(X_f) &= \frac{1}{4}[1+m_1+1-m_1+1+m_2+1-m_2](1-\lambda(t)) \\ &\quad +\frac{\lambda(t)}{2}[1+1] \\ &= 1 \end{aligned}$$

Now, we compute the two conditions required to determine m_1 and m_2. We use the expectation of the equity process to determine one equation. We exploit the fact that under risk-neutrality the equity return must equal the risk free rate of interest. This leads to the following:

$$\begin{aligned} E\left[\frac{S(t+h)}{S(t)}\right] &= \frac{1}{4}(1-\lambda(t))[a(t)(1+m_1)+b(t)(1-m_1) \\ &\quad +a(t)(1+m_2)+b(t)(1-m_2)] + \frac{\lambda(t)}{2}[0] \qquad (18) \\ &= \exp[r(t)h] \end{aligned}$$

Hence the stock return is set equal to the riskfree return. This implies the following from a simplification of equation (18):

$$m_1 + m_2 = \frac{\frac{4e^{r(t)h}}{1-\lambda(t)} - 2[a(t)+b(t)]}{a(t)-b(t)} \equiv A \qquad (19)$$

Our second condition comes from the correlation specification. Let the correlation (coincident with covariance for unit valued variables) between the shocks $[X_f(t), X_s(t)]$ be equal to ρ, where $-1 \leq \rho \leq 1$. A simple calculation follows (ignoring the branches of default, since the correlation in that case is undefined):

$$\begin{aligned} Cov[X_f(t), X_s(t)] &= \frac{1}{4}(1-\lambda(t))[1+m_1-1+m_1-1-m_2+1-m_2] \\ &= \frac{m_1-m_2}{2}(1-\lambda(t)). \qquad (20) \end{aligned}$$

Setting this equal to ρ, we get the equation

$$m_1 - m_2 = \frac{2\rho}{1-\lambda(t)} \equiv B. \tag{21}$$

Solving the two equations (19) and (21) leads to the following solution:

$$m_1 = \frac{A+B}{2}, \quad m_2 = \frac{A-B}{2} \tag{22}$$

These values may now be substituted into the probability measure in Table 1. Notice that since the interest rate $r(t)$ only enters the probabilities and not the random shock $X_s(t), \forall t$, the equity lattice will also be recombining, just as was the case with the HJM model for the term structure. Hence, the product space of defaultable equity and interest rates will also be recombining. As interest rates change, the probability measure will also change, but this will not impact the recombining property of the lattice. Finally, note that the analysis here is valid irrespective of the properties of the equity process chosen, and applied, in particular to any choice of CEV coefficient in the models we consider.

A.1 Ensuring a valid probability measure

It is also necessary that the solutions for m_1 and m_2 be such that the resultant probabilities do not become negative or greater than 1. From Table 4, we see that the necessary condition is $-1 \le m_i \le +1, \quad i = 1, 2$. To see this, note that the greatest absolute value of the probabilities on the non-defaultable branches is when $\lambda = 0$. Given this, we require the following 2 conditions on m_1, so as to be valid probabilities:

$$0 \le \frac{1}{4}[1+m_1] \le 1, \quad 0 \le \frac{1}{4}[1-m_1] \le 1$$

which implies that $-1 \le m_1 \le +1$. The same condition is derived for m_2.

Of course, the preceding analysis really implies that there is a range for the value of default probability $\lambda(t)$ which is consistent with the equity and term structure processes. Hence, we can derive the corresponding values of $\lambda(t)$ that correspond to the permissible ranges for m_1, m_2 above. This results in 8 bounds, which are presented in the Table 4. After computing the value of the default probability we check that it satisfies these bounds, else we set it to be within the range values.

Table 4: Bounds on default probabilities. These 8 conditions specify limit values on one period default probabilities $\lambda(t)$. Some of these conditions may not apply in the sense that they suggest negative limits for probabilities, which are superseded by the lower limit condition of zero value. The upper bound on $\lambda(t)$ will be the minimum of the positive limits in the table.

Condition on m_i	Limit Value of $\lambda(t)$
$0 \leq \frac{1}{4}[1+m_1]$	$\frac{1}{2b(t)}[-2e^{r(t)h} - a(t)\rho + b(t)(\rho+2)]$
$\frac{1}{4}[1+m_1] \leq 1$	$\frac{1}{4a(t)-2b(t)}[-2e^{r(t)h} - a(t)(\rho-4) + b(t)(\rho-2)]$
$0 \leq \frac{1}{4}[1-m_1]$	$\frac{1}{2a(t)}[-2e^{r(t)h} - a(t)(\rho-2) + b(t)\rho]$
$\frac{1}{4}[1-m_1] \leq 1$	$\frac{1}{2a(t)-4b(t)}[2e^{r(t)h} + a(t)(\rho+2) - b(t)(\rho+4)]$
$0 \leq \frac{1}{4}[1+m_2]$	$\frac{1}{2b(t)}[-2e^{r(t)h} - b(t)(\rho-2) + a(t)\rho]$
$\frac{1}{4}[1+m_2] \leq 1$	$\frac{1}{4a(t)-2b(t)}[-2e^{r(t)h} - b(t)(\rho+2) + a(t)(\rho+4)]$
$0 \leq \frac{1}{4}[1-m_2]$	$\frac{1}{2a(t)}[-2e^{r(t)h} - b(t)\rho + a(t)(\rho+2)]$
$\frac{1}{4}[1-m_2] \leq 1$	$\frac{1}{2a(t)-4b(t)}[2e^{r(t)h} + b(t)(\rho-4) - a(t)(\rho-2)]$

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